JORDAN TRIPLE SYSTEMS : PROPERTIES OF THE GENERIC MINIMAL POLYNOMIAL

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Abstract

Hermitian positive Jordan triple systems (HPJTS) correspond to bounded symmetric complex domains. We give a survey of the properties of HPJTS and especially of their generic minimal polynomial. As an application, we give a description of the canonical projective realization of the compactification. It turns out that for a natural normalization, the Euclidean volume of a bounded circled homogeneous complex domain is an integer which is equal to the projective degree of the above compactification. The properties of the generic minimal polynomial also lead to the concept of *polynomial morphisms of JTS, for which some open problems are stated.*

1 Hermitian positive Jordan triple systems

Definition 1 *Let V be a finite dimensional complex vector space. A structure of* Jordan triple system over V is a ternary product (the Jordan triple product) $(x,y,z)\mapsto\{xyz\}=D(x,y)z=Q(x,z)y$ which is complex bilinear and symmetric with respect to (x,z), complex antilinear with respect to y and satisfies the Jordan identity

$$\{xy\{uvw\}\}-\{uv\{xyw\}\}=\{\{xyu\}vw\}-\{u\{vxy\}w\}.$$

A Jordan triple system is said hermitian positive if $(u|v)=\operatorname{tr}D(u,v)$ is positive definite.

The quadratic representation

is defined by

$$Q(x)y=\frac{1}{2}\{xyx\}.$$

The following fundamental identity for the quadratic representation is a consequence of the Jordan identity :

$$Q(Q(x)y)=Q(x)Q(y)Q(x).$$

The Bergman operator B is defined by

$$B(x,y)=I-D(x,y)+Q(x)Q(y),$$

where I denotes the identity operator in V. It is also a consequence of the Jordan identity that the following fundamental identity holds for the Bergman operator :

$$Q(B(x,y)z)=B(x,y)Q(z)B(y,x).$$

Example 1 *Type $I_{p,q}$, $1\le p\le q$: $V=\mathcal{M}_{\sqrt{},\mathrm{II}}(\mathbb{C})$ (space of $p\times q$ matrices with complex entries), endowed with the triple product*

$$\{xyz\}=x{}^t\overline{y}z+z{}^t\overline{y}x.$$

Example 2 *Type II_n : $V=\mathcal{A}_\backslash(\mathcal{C})$ (space of $n\times n$ alternating matrices) with the same triple product.*

Example 3 *Type* III_n : $V = \mathcal{S}_\backslash(C)$ *(space of* $n \times n$ *symmetric matrices) with the same triple product.*

Example 4 *Type* IV_n : $V = C^n$ *with the quadratic operator defined by*

$$Q(x)y = q(x, \overline{y})x - q(x)\overline{y},$$

where $q(x) = \sum x_i^2$, $q(x, y) = 2\sum x_i y_i$.

Example 5 *Type* $VI : V = \mathcal{H}_\ni(\mathcal{O}_C)$, *the space of* 3×3 *matrices with entries in the space* O_C *of octonions over* C, *which are hermitian with respect to the Cayley conjugation ; the quadratic operator is defined by*

$$Q(x)y = (x|y)x - x^\sharp \times \overline{y},$$

where $\times$ *denotes the Freudenthal product,* $x^\sharp$ *the adjoint matrix in* $\mathcal{H}_\ni(\mathcal{O}_C)$ *and* $(x|y)$ *the standard hermitian product in* $\mathcal{H}_\ni(\mathcal{O}_C)$.

Example 6 *Type* $V : V = \mathcal{M}_{\in,\infty}(\mathcal{O}_C)$, *the subspace of* $\mathcal{H}_\ni(\mathcal{O}_C)$ *consisting in matrices of the form*

$$\begin{pmatrix} 0 & a_3 & \tilde{a}_2 \\ \tilde{a}_3 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}$$

with the same quadratic operator. Here $\tilde{a}$ *denotes the Cayley conjugate of* $a \in O_C$.

Actually, the preceding examples exhaust the list of simple Hermitian positive Jordan triple systems.

2 The generic minimal polynomial

2.1 The generic minimal polynomial of a unital Jordan algebra

Recall that a (finite dimensional) Jordan algebra over C is a complex vector space A with a commutative bilinear product $(x, y) \longmapsto xy$ satisfying the Jordan identity

$$x(x^2y) = x^2(xy).$$

It is known that a Jordan algebra is power associative. Assume that the Jordan algebra A has a unit element e. Then for each $x \in A$ and each polynomial $p \in C[T]$, it is possible to define $p(x)$ in the usual way ; the map $p \longmapsto p(x)$ is then an algebra homomorphism, whose kernel $\mathcal{I}_\S$ is an ideal of $C[T]$. The monic generator m_x of $\mathcal{I}_\S$ is the minimal polynomial of x ; its degree is the rank of x. An element $x \in A$ is called regular if its rank is maximal ; the rank of regular elements is called the rank of the Jordan algebra A. The following result holds in a finite dimensional Jordan algebra with unit element (and more generally in a power associative algebra) :

Proposition 1 *There exist polynomials* $m_1, \ldots, m_r$ *on* A , *of respective degrees* $1, \ldots, r$, *such that, for each regular element* $x \in A$, *the minimal polynomial* m_x *is equal to*

$$m_x = T^r - m_1(x)T^{r-1} + \cdots + (-1)^r m_r(x).$$

Definition 2 *The polynomial*

$$m(T, x) = T^r - m_1(x)T^{r-1} + \cdots + (-1)^r m_r(x)$$

is called the generic minimal polynomial of A at x. The linear form m_1 is called the generic trace and the polynomial m_r is called the determinant and denoted $m_r = \det$.

The following relations hold :

$$m(\lambda, x) = \det(\lambda e - x), \ \det e = 1, \ \det(P(x)y) = (\det x)^2 \det y.$$

Here P denotes the quadratic operator of the Jordan algebra A :

$$P(x)y = 2x(xy) - x^2y.$$

2.2 The generic minimal polynomial of a Jordan algebra

Here we consider a Jordan algebra A over C, but we don't further assume it has a unit element. Let $\tilde{A} = C \oplus A$ the algebra obtained from A in the usual way : $(\lambda \oplus x)(\mu \oplus y) = \lambda\mu \oplus (\lambda y + \mu x + xy)$. It is easy to check that $\tilde{A}$ is a Jordan algebra with unit element $\tilde{e} = 1 \oplus 0$. The generic minimal polynomial $\tilde{m}$ of $0 \oplus x$ in $\tilde{A}$ has the form $\tilde{m}(T, 0 \oplus x) = Tm(T, x)$. If A has already a unit element, then $m(T, x)$ coincides with the previously defined generic minimal polynomial.

In the general case, the relation $\tilde{m}(T, 0 \oplus x) = Tm(T, x)$ will be taken as the definition of the generic minimal polynomial $m(T, x)$.The generic norm is $N(x) = m(1, x)$. One has $\widetilde{\det}(\lambda \oplus -x) = \lambda m(\lambda, x)$ and $\widetilde{\det}(1 \oplus -x) = N(x)$, where $\widetilde{\det}$ denotes the determinant in $\tilde{A}$. An element $x \in A$ is said to be regular if $0 \oplus x$ is regular in $\tilde{A}$, which is equivalent to say that $\langle x, x^2, \ldots, x^{r+1} \rangle$ has maximal dimension r.

2.3 The generic minimal polynomial of a Jordan triple system

Let us consider now a Jordan triple system V. If $y \in V$, a structure of Jordan algebra, denoted $V^{(y)}$, is defined by taking on V the product $\circ_y$ defined by

$$x \circ_y z = \frac{1}{2}\{xyz\}.$$

Then $V^{(y)}$ is a Jordan algebra. We say that $(x, y) \in V \times V$ is regular if $V^{(y)}$ has maximal rank and if x is regular in $V^{(y)}$. The number $r = \max\{\rho(y); y \in V\}$, where $\rho(y)$ denotes the rank of the Jordan algebra $V^{(y)}$, is called the rank of the Jordan triple system V.

Theorem 1 *Let V be a Jordan triple system of rank r. There exist (real) polynomials $m_1, \ldots, m_r$ on $V \times \overline{V}$, homogeneous of respective bidegrees $(1,1), \ldots, (r, r)$, such that for each regular pair $(x, y) \in V \times V$, the minimal polynomial of x in $V^{(y)}$ is equal to*

$$T^r - m_1(x, y)T^{r-1} + \cdots + (-1)^r m_r(x, y).$$

Here $\overline{V}$ denotes the space V with the conjugate complex structure.

Definition 3 *The polynomial*

$$m(T, x, y) = T^r - m_1(x, y)T^{r-1} + \cdots + (-1)^r m_r(x, y)$$

is called the generic minimal polynomial of V (at (x, y)). The (inhomogeneous) polynomial $N : V \times \overline{V} \to C$ defined by

$$N(x, y) = m(1, x, y)$$

is called the generic norm.

Proposition 2 *The following identities hold for the polynomials m_k $(1 \le k \le r)$:*

$$m_k(x, y) = \overline{m_k(y, x)},$$
$$m_k(Q(x)y, z) = \overline{m_k(x, Q(y)z)}.$$

Hereunder are the minimal polynomials corresponding to some examples listed in the preceding section :

1. Type $\mathrm{I}_{p,q}$: $m(T, x, y) = \mathrm{Det}\,(TI_p - xy^*)$, where Det is the usual determinant of square matrices.
2. Type III_n : $m(T, x, y) = \mathrm{Det}\,(TI_n - xy^*)$.
3. Type IV_n : $m(T, x, y) = T^2 - q(x, \overline{y}) + q(x)q(\overline{y})$.

3 Inverse and quasi-inverse

3.1 Invertibility in a Jordan algebra with unit element

Let A be a Jordan algebra with unit element e. Recall that the quadratic operator P in A is defined by $P(x)y = 2x(xy) - x^2y$. For $a \in A$, we denote by $C[a]$ the unital subalgebra generated by a.

Theorem 2 *For $a \in A$, the following properties are equivalent :*

1. *a is invertible in $C[a]$;*
2. *$P(a)$ is invertible ;*
3. *e is in the range of $P(a)$;*
4. *there exists $b \in A$ such that $P(a)b = a$ and $P(a)b^2 = e$;*
5. $\det a \neq 0$.

If an element $a \in A$ satisfies to these equivalent conditions, then it is called invertible in the Jordan algebra A ; if a is invertible, the element $a^{-1} = P(a)^{-1}a$ is called the inverse of a. Note that $ab = e$, but this last condition is in general not sufficient for a to be invertible in the above sense.

3.2 Quasi-invertibility in a Jordan algebra

Let A be a Jordan algebra and let $\tilde{A} = C \oplus A$ the corresponding unital algebra. For $x \in A$, we denote by $L(x)$ the multiplication operator : $L(x)y = xy$.

Definition 4 *An element $x \in A$ is called* quasi-invertible iff $1 \oplus -x$ is invertible in the unital Jordan algebra $\tilde{A}$. If $x \in A$ is quasi-invertible, the quasi-inverse of x is the element z such that $1 \oplus z$ is the inverse of $1 \oplus -x$ in $\tilde{A}$.

Theorem 3 *For $x \in A$, the following properties are equivalent :*

1. *x is quasi-invertible ;*
2. *there exists $z \in C_0[x]$ (z is the value at x of a polynomial without constant coefficient) such that $z - x = xz$;*
3. *the linear operator $I - 2L(x) + P(x)$ is invertible on A ;*
4. *$2x - x^2$ is in the range of $I - 2L(x) + P(x)$;*
5. *there exists $z \in A$ such that $(I - 2L(x) + P(x))z = x - x^2$ and $(I - 2L(x) + P(x))z^2 = x^2$;*
6. *$N(x) \neq 0$.*

If x is quasi-invertible, its quasi-inverse is given by

$$z = (I - 2L(x) + P(x))^{-1}x^2;$$

if x is close to 0, one also has $z = \sum_{n=1}^{\infty} x^n$.

3.3 Quasi-invertibility in a Jordan triple system

Let V be a Jordan triple system. For $y \in V$, recall that $V^{(y)}$ is the Jordan algebra obtained by taking on V the product $x \circ_y z = \frac{1}{2}\{xyz\}$. The operators L and P for $V^{(y)}$ turn out to be $L_y(x) = \frac{1}{2}D(x, y)$ and $P_y(x) = Q(x)Q(y)$. As a consequence, $I - 2L_y(x) + P_y(x) = B(x, y)$, the Bergman operator.

Definition 5 *A pair $(x, y) \in V \times V$ is called* quasi-invertible iff x is quasi-invertible in the Jordan algebra $V^{(y)}$. The quasi-inverse x^y is the quasi-inverse of x in V.

Theorem 4 *For $(x, y) \in V \times V$, the following conditions are equivalent :*

1. *the pair (x,y) is quasi-invertible ;*

2. *the Bergman operator $B(x,y)$ is invertible ;*

3. *there exists $z \in V$ such that $B(x,y)z = x - Q(x)y$ and $B(x,y)Q(z)y = Q(x)y$;*

4. *the element $2x - Q(x)y$ belongs to the range of $B(x,y)$;*

5. *$N(x,y) \neq 0$.*

If (x,y) is quasi-invertible, the quasi-inverse x^y is given by

$$x^y = B(x,y)^{-1}(x - Q(x)y).$$

If x is close to 0, then $x^y = \sum_{n=1}^{\infty} x^{(n,y)}$, where $x^{(n,y)}$ denotes the n-th power of x in the Jordan algebra $V^{(y)}$.

Quasi-invertibility has the following symmetry property :

Proposition 3 *A pair (x,y) is quasi-invertible iff (y,x) is quasi-invertible ; then $y^x = y + Q(y)x^y$.*

4 Spectral theory

¿From now on, V is a hermitian positive Jordan triple system. Moreover, we will assume that V is simple, that is V is not the direct sum of two non trivial subsystems with component-wise triple product. Any hermitian positive Jordan triple system is in fact semi-simple, that is the direct sum of a finite family of simple subsystems.

An automorphism $f : V \to V$ of the Jordan triple system V is a complex linear isomorphism preserving the triple product : $f\{u,v,w\} = \{fu, fv, fw\}$. The automorphims of V form a group, denoted Aut V, which is a compact Lie group ; we will denote by K its identity component.

Definition 6 *An element $c \in V$ is called* tripotent *if c is an idempotent in $V^{(c)}$, that is if $\{ccc\} = 2c$.*

Proposition 4 *If c is a tripotent, the operator $D(c,c)$ annihilates the polynomial $T(T-1)(T-2)$.*

Definition 7 *Let e be a tripotent. The decomposition $V = V_0(c) \oplus V_1(c) \oplus V_2(c)$, where $V_j(c)$ is the eigenspace $V_j(c) = \{x \in V \; ; \; D(c,c)x = jx\}$, is called the* Peirce decomposition of V (with respect to the tripotent c).

Definition 8 *Two tripotents c_1 and c_2 are called* orthogonal if $D(c_1,c_2) = 0$.

If c_1 and c_2 are orthogonal tripotents, then $D(c_1,c_1)$ and $D(c_2,c_2)$ commute and $c_1 + c_2$ is also a tripotent.

Definition 9 *A non zero tripotent c is* primitive if it is not the sum of non zero orthogonal tripotents. A tripotent c is maximal if there is no non zero tripotent orthogonal to c. A frame of V is a maximal sequence $(c_1, \ldots, c_r)$ of pairwise orthogonal primitive tripotents.

Definition 10 *Let $\mathbf{c} = (\mathbf{c_1}, \ldots, \mathbf{c_r})$ be a frame. For $0 \leq i \leq j \leq r$, let*

$$V_{ij}(\mathbf{c}) = \left\{\mathbf{x} \in \mathbf{V} \; ; \; \mathbf{D}(\mathbf{c_k}, \mathbf{c_k})\mathbf{x} = (\delta_\mathbf{i}^\mathbf{k} + \delta_\mathbf{j}^\mathbf{k})\mathbf{x}, \; \mathbf{1} \leq \mathbf{k} \leq \mathbf{r}\right\} :$$

the decomposition $V = \bigoplus_{0 \leq i \leq j \leq r} V_{ij}(\mathbf{c})$ is called the simultaneous Peirce decomposition with respect to the frame $\mathbf{c}$.

Theorem 5 *Let V be a simple hermitian positive Jordan triple system. Then there exist frames for V. All frames have the same number of elements, which is equal to the rank r of V. The subspaces $V_{ij} = V_{ij}(\mathbf{c})$ of the simultaneous Peirce decomposition have the following properties : $V_{00} = 0$; $V_{ii} = \mathbb{C}e_i$ $(0 < i)$; all V_{ij}'s $(0 < i < j)$ have the same dimension ; all V_{0i}'s $(0 < i)$ have the same dimension. The group K acts transitively on the manifold $\mathcal{F}$ of frames of V.*

Definition 11 *The* numerical invariants of V are

$$a = \dim V_{ij} \quad (0 < i < j),$$

$$b = \dim V_{0i} \quad (0 < i).$$

The genus of V is the number defined by

$$g = 2 + a(r-1) + b.$$

Theorem 6 *Let V be a simple hermitian positive Jordan triple system. Then any $x \in V$ can be written in a unique way*

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p,$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and $c_1, c_2 \ldots, c_p$ are pairwise orthogonal tripotents. The element x is regular iff $p = r$; then $(c_1, c_2 \ldots, c_r)$ is a frame of V.

Definition 12 *The decomposition $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$ is called* spectral decomposition of x.

The following proposition shows the relation between the generic minimal polynomial and the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of a regular element x.

Proposition 5 *Let V be a simple hermitian positive Jordan triple system. Let $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$ be the spectral decomposition of a regular element x. Then the generic minimal polynomial at (x, x) is*

$$m(T, x, x) = \prod_{i=1}^{r} (T - \lambda_i^2).$$

The following identities hold :

$$\begin{aligned} \det B(x,y) &= N(x,y)^g, \\ \operatorname{tr} \mathrm{D}(\mathrm{x},\mathrm{y}) &= g\, m_1(x,y). \end{aligned}$$

5 Schmid decomposition

Let V be a simple hermitian positive Jordan triple system. We denote by $\mathcal{P}(\mathcal{V})$ the space of complex polynomials on V. The group K acts naturally on $\mathcal{P}(\mathcal{V})$.

Definition 13 *For $\mathbf{n} = (\mathbf{n_1}, \mathbf{n_2}, \ldots, \mathbf{n_r}) \in \mathbf{N^r}$ with $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, let $m_{\mathbf{n}}$ be the unique complex polynomial on $V \times \overline{V}$ such that, for each regular x with spectral decomposition $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$, the value of $m_{\mathbf{n}}$ at (x, x) is*

$$m_{\mathbf{n}}(x,x) = \sum \lambda_{i_1}^{2n_1} \lambda_{i_2}^{2n_2} \ldots \lambda_{i_r}^{2n_r}.$$

For $y \in V$, we denote by $m_{\mathbf{n},\mathbf{y}}$ the polynomial on V defined by

$$m_{\mathbf{n},\mathbf{y}}(x) = m_{\mathbf{n}}(x, y)$$

and by $\mathcal{P}_{\mathbf{n}}(\mathcal{V})$ the subspace of $\mathcal{P}(\mathcal{V})$ spanned by the polynomials $m_{\mathbf{n},\mathbf{y}}$, $y \in V$.

Remark 1 *Let $1 \leq k \leq r$; let us denote by $< k >$ the multi-integer $< k > = (\overbrace{1, \ldots 1}^{k \text{ times}}, 0, \ldots, 0)$. Then $m_{<k>} = m_k$, the homogeneous component of bidegree (k, k) of the generic norm.*

Theorem 7 *(Schmid decomposition) The space $\mathcal{P}(\mathcal{V})$ admits the following decomposition into irreducible, pairwise inequivalent K-modules :*

$$\mathcal{P}(\mathcal{V}) = \bigoplus_{\mathbf{n} \geq 0} \mathcal{P}_{\mathbf{n}}(\mathcal{V}),$$

where $\mathbf{n} \geq 0$ for $\mathbf{n} = (\mathbf{n_1}, \mathbf{n_2}, \ldots, \mathbf{n_r}) \in \mathbf{N^r}$ means $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$. Moreover, the polynomial $m_{\mathbf{n}}$ is a reproducing kernel for $\mathcal{P}_{\mathbf{n}}(\mathcal{V})$, which means there is an hermitian scalar product $(\,|\,)$ on $\mathcal{P}_{\mathbf{n}}(\mathcal{V})$ such that

$$m_{\mathbf{n}}(x, y) = (m_{\mathbf{n},\mathbf{x}} | m_{\mathbf{n},\mathbf{y}}).$$

Let us consider the (antilinear) isomorphism $\vartheta : V \to V^*$ defined by

$$\vartheta(y)(x) = m_1(x,y) = g^{-1} \operatorname{tr} \mathrm{D(x,y)}.$$

Using this isomorphism, we get an antilinear isomorphism $\varphi : \mathcal{P}(\mathcal{V}) \longrightarrow \bigodot_* \mathcal{V}$, where $\bigodot_* V$ is the symmetric algebra of V. For $\mathbf{n} \geq 0$, let $V_\mathbf{n} = \varphi(\mathcal{P}_\mathbf{n}(\mathcal{V}))$ and $\sigma_\mathbf{n} : V \longrightarrow V_\mathbf{n} \subset \bigodot_{|\mathbf{n}|} V$ defined by $\sigma_\mathbf{n}(y) = \varphi \circ \overline{m_{\mathbf{n},\mathbf{y}}}$ $(y \in V)$. The hermitian scalar product $(\,|\,)$ is also transferred from $\mathcal{P}_\mathbf{n}(\mathcal{V})$ to $V_\mathbf{n}$; the following relation then holds :

$$m_\mathbf{n}(x,y) = (\sigma_\mathbf{n}(x)|\sigma_\mathbf{n}(y)).$$

Note that $\sigma_\mathbf{0}(x) = 1$ and that $\sigma_{<1>} : V \longrightarrow V$ is the identity map.

6 Compactification

6.1 Compactification of an hermitian positive Jordan triple system

The following description of the compactification has been given by O. Loos. Let V be an HPJTS ; one can define an equivalence relation $\sim$ on $V \times V$ by

$$(x,y) \sim (x',y') \Longleftrightarrow ((x, y-y')\mathrm{isquasi-invertible\&}\ \mathrm{x}' = \mathrm{x}^{\mathrm{y}-\mathrm{y}'}).$$

Roughly speaking, this equivalence relation means $x^y = x'^{y'}$. Let us denote by $[x,y]$ the equivalence class of (x,y) and by X the quotient space $(V \times V)/\sim$.

Theorem 8 *There exists a unique structure of smooth algebraic variety on X such that, for each $v \in V$, $X_v = \{[x,v]\,;\, x \in V\}$ is an open affine subvariety and $[x,v] \mapsto x$ is an isomorphism of X_v on V.*

The injection $x \mapsto [x,0]$ of V in X will be called the canonical compactification of V.

6.2 Projective imbedding

Let V be a simple HPJTS of rank r. For $0 \leq k \leq r$, we write V_k for $V_{<k>}$ and σ_k for $\sigma_{<k>}$. Let W be the subspace of $\bigodot_* V$ defined by

$$W = \oplus_{0 \leq k \leq r} V$$

and $\sigma : V \longrightarrow P(W)$ the map given by

$$\sigma(x) = [1, x, \sigma_2(x), \ldots, \sigma_r(x)].$$

The compactification described above has then the following projective realization.

Theorem 9 *Let V be a simple HPJTS of rank r. The closure of $\sigma(V)$ is an algebraic submanifold $\widetilde{X}$ of $P(W)$, which is isomorphic to X, and $\sigma : V \longrightarrow \widetilde{X}$ is isomorphic to the canonical compactification $V \longrightarrow X$.*

¿From now on, we will identify X with $\widetilde{X}$.

6.3 Volume computations

Let α be the Kähler form on V associated to the hermitian inner product m_1 :

$$\alpha = \frac{i}{2\pi} \partial\overline{\partial} m_1.$$

We endow V with the volume form α^n $(n = \dim V)$; the volume of the unit ball associated to m_1 is then equal to 1.

On $W = \oplus_{0 \leq k \leq r} V$, we put the hermitian scalar product $(\,|\,)$ which is the direct sum of the hermitian scalar products $(\,|\,)$ on the V_k's. Let F be defined on W by $F(z) = (z \mid z)$ and let β be the associated Fubini-Study form on $P(W)$:

$$\beta = \frac{i}{2\pi} \partial\overline{\partial} \log F.$$

For a smooth submanifold Z of pure dimension d in $P(W)$, the degree of Z in $P(W)$ is then given by

$$\deg Z = \int_Z \beta^d.$$

In particular, the degree in $P(W)$ of the compactification X of V is

$$\deg X = \int_X \beta^n = \int_V \sigma^* \beta^n,$$

as $\sigma(V)$ is an open dense subset of X. From the definition of σ and from the relations $m_k(x,x) = (\sigma_k(x) \mid \sigma_k(x))$, it follows that

$$\sigma^*\beta = \frac{i}{2\pi}\partial\overline{\partial}\log\left(1 + m_1(x,x) + \cdots + m_r(x,x)\right) = \frac{i}{2\pi}\partial\overline{\partial}\log N(x,-x).$$

The following proposition allows us to compute $\sigma^*\beta^n$ (where $n = \dim V$).

Proposition 6 *The identity*

$$\left(\frac{i}{2\pi}\partial\overline{\partial}\log N(x,-x)\right)^n = \det B(x,x)^{-1}\,\alpha^n = N(x,x)^{-g}\alpha^n$$

holds in the hermitian positive Jordan triple system V.

Proposition 7 *Let*

$$\Phi : \mathcal{F} \times \{\lambda_\infty > \lambda_\in > \cdots > \lambda_\nabla > \imath\} \longrightarrow V_{\nabla]\}}$$

be the diffeomorphism defined by

$$\Phi((c_1,\ldots,c_r),(\lambda_1,\ldots,\lambda_r)) = \sum_{j=1}^{r} \lambda_j c_j;$$

here $\mathcal{F}$ is the manifold of frames of V and V_{reg} the open dense subset of regular elements of V. The pull-back of the volume element α^n by Φ is

$$\Phi^*\alpha^n = \Theta \wedge \prod_{j=1}^{r} \lambda_j^{2b+1} \prod_{1\leq k \leq j \leq r} \left(\lambda_j^2 - \lambda_k^2\right)^a \, \mathrm{d}\lambda_1 \wedge \ldots \mathrm{d}\lambda_r,$$

where a and b are the numerical invariants of V and Θ is a K-invariant volume form on V.

Definition 14 *Let V be an HPJTS. The* bounded symmetric domain D associated to V is the unit ball of V for the spectral norm, that is the set of elements $x \in V$ whose spectral decomposition is

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$$

with $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$. We define the real analytic maps $\vartheta : V \longrightarrow D$ and $\psi : D \longrightarrow V$, inverse of each other, by

$$\vartheta(x) = B(x,-x)^{-1/4}x = \sum \lambda_j(1+\lambda_j^2)^{-1/2}c_j$$

if $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p \in V$ and

$$\psi(x) = B(x,x)^{-1/4}x = \sum \lambda_j(1-\lambda_j^2)^{-1/2}c_j$$

if $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p \in D$.

Theorem 10 *The following relations hold in a simple HPJTS V of dimension n and genus g :*

$$\vartheta^*\left(N(x,-x)^s\alpha^n\right) = N(x,x)^{-g-s}\alpha^n,$$
$$\psi^*\left(N(x,x)^s\alpha^n\right) = N(x,-x)^{-g-s}\alpha^n.$$

Taking $s = 0$ in the last relation and integrating over V, we get

Theorem 11 *The volume of the bounded symmetric domain D (with respect to the normalized form α^n) is equal to the degree of the compactification X in $P(W)$.*

7 Open problems

1. If one looks over the examples given by the classification, one can see that the V_k's $(1 \leq k \leq r)$ have a structure of Jordan triple system, such that $\sigma_k : V \longrightarrow V_k$ *is homogeneous of degree* k *and is* a polynomial morphism of Jordan triple systems, that is satisfies identically

$$\sigma_k(Q(x)y) = Q_k(\sigma_k(x))\,\sigma_k(y),$$

   where Q_k is the quadratic operator for V_k. Understand this JTS structure on V_k?

2. There are other polynomial morphisms of Jordan triple systems, for example the quadratic representation

$$Q : V \longrightarrow End_R(V)$$

   which is a quadratic morphism as shown by the fundamental identity

$$Q(Q(x)y) = Q(x)Q(y)Q(x).$$

   Understand and classify homogeneous polynomial morphisms between simple hermitian positive Jordan triple systems?

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